Olympiad Number Theory Through Challenging Problems

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FIFTH EDITION

Mathematics is the queen of sciences and number theory is the queen of mathematics.

– Carl Friedrich Gauss

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CHAPTER 1

THE ART OF PROOFS

§1.1 Well-Ordering Principle

The set of ***integers*** are $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. The properties of addition $(+)$ and multiplication $(\cdot)$ for integers a and b include:

(I) Closure: $a + b$ and $a \cdot b$ are both integers.

(II) Associativity: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(III) Commutativity: $a + b = b + a$ and $a \cdot b = b \cdot a$.

(IV) Identity: $a + 0 = a$ and $a \cdot 1 = a$.

(V) Distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

(VI) Additive Inverse: $a + (-a) = 0$.

Positive integers are $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. The additive inverses of the positive integers are the negative integers. The natural numbers, $\mathbb{N}_0$, consist of zero combined with the positive integers. They are equipped with an ordering relation; we write $a < b$ if $b - a$ is positive.

Example 1.1. Prove that $\min(a, b) + \max(a, b) = a + b$.

Proof. We have two cases to consider. If $a \leq b$, then we have $\min(a, b) = a$ and $\max(a, b) = b$. Otherwise, if $b < a$, then $\min(a, b) = b$ and $\max(a, b) = a$. Either way, the result holds. □

Axiom (Well-Ordering). Every non-empty subset of $\mathbb{Z}^+$ has a least element.

The well-ordering principle serves as a starting block from which we build number theory.

Definition. $x \in S$ denotes "x belongs to set S" and $R \subset S$ denotes "R is a subset of S".

Example 1.2. Prove that there is no integer between 0 and 1.

Proof. Assume for the sake of contradiction that $S = \{c \in \mathbb{Z} \mid 0 < c < 1\}$, the set of integers between 0 and 1, is not empty. Hence, S must have a smallest element, say m. However, we see that $m^2 \in \mathbb{Z}$ from closure over multiplication and $0 < m^2 < m < 1$. This contradicts the minimality of m, hence $S = \emptyset$ and there are no integers between 0 and 1. □

A ***rational number*** can be expressed as a/b where a and b are integers and $b \neq 0$. The rationals, $\mathbb{Q}$, are a field since all nonzero elements have a multiplicative inverse. They can be formally defined as an equivalence class of pairs of integers (a, b) with $b \neq 0$ and equivalence relation $(a_1, b_1) \sim (a_2, b_2)$ if and only if $a_1 b_2 = a_2 b_1$.

Example 1.3. Prove that the rational numbers are not well-ordered.

Proof. We must show that there is no lower bound to the rationals. Consider the set

$$S = \left\{ \frac{1}{n} \text{ for } n \in \mathbb{Z}^+ \right\}.$$

As n grows larger the set approaches 0. However, this lower bound is never reached. □

An ***irrational number*** cannot be expressed as the ratio of two integers. Around 500 BC, Pythagoras founded a religion called Pythagoreanism. The followers thought numbers explained everything in life, from nature to music. According to legend, Hippasus was a Pythagorean who was an excellent mathematician. While looking at the pentagram, he took measures of the length of several sides and found the ratio to be an irrational number, the golden ratio.

Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities: $(a+b)/a = a/b \overset{\text{def}}{=} \varphi$. Letting $\varphi = a/b$ in the equation, we see

$$1 + \frac{1}{\varphi} = \varphi \implies \varphi^2 - \varphi - 1 = 0.$$

Using the quadratic formula, $\varphi = \frac{1+\sqrt{5}}{2}$. The other root is $\psi = \frac{1-\sqrt{5}}{2}$.

Example 1.4. Prove that $\sqrt{2}$ is irrational.

Proof by Contradiction. For positive integers a and b, let $\sqrt{2} = \dfrac{a}{b}$. Consider the set

$$X = \{k\sqrt{2} : \text{both } k \text{ and } k\sqrt{2} \text{ are positive integers}\}.$$

Since $a = b\sqrt{2}$, X is not empty. Let the smallest element of X be $m = n\sqrt{2}$. Consider

$$m\sqrt{2} - m = m\sqrt{2} - n\sqrt{2} = (m-n)\sqrt{2}.$$

Since $m\sqrt{2} - m = 2n - m$ is a positive integer, we have $m\sqrt{2} - m \in X$. However, $m\sqrt{2} - m$ is an element of X that is less than m, contradiction. Therefore X is empty and $\sqrt{2}$ is irrational. □

The ***reals***, $\mathbb{R}$, consist of all rational and irrational numbers, therefore $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

§1.2 Induction

Induction is a popular proof technique used in mathematics. We begin with an example.

Example 1.5. Prove the identity $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$.

Proof. We begin by testing the identity for small values of n:

$$1 = 2 - 1, \quad 1 + 2 = 4 - 1, \quad 1 + 2 + 4 = 8 - 1.$$

We now assume the identity is true for an arbitrary $n = k$:

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1. \tag{Hypothesis}$$

Adding the next power of 2 to both sides of our assumption:

$$\begin{aligned}\left[1 + 2 + 2^2 + \cdots + 2^k\right] + 2^{k+1} &= \left[2^{k+1} - 1\right] + 2^{k+1} \\ &= 2^{k+2} - 1.\end{aligned}$$

Therefore, we have proven that if the identity is true for $n = k$, then it is also true for $n = k + 1$. Imagining the natural numbers as dominoes, we knock down the first domino ($n = 0$) and every domino knocks down the next one. Therefore, the identity is true for all natural numbers n. □

Principle (Mathematical Induction). To prove a statement P for all positive integers at least n_0,

(1) **Base Case:** Show $P(n_0)$.

(2) **Inductive Step:** Show $P(k)$ implies $P(k + 1)$ for any positive integer $k \geq n_0$.

The assumption we make in the inductive step is known as the induction hypothesis.

Proof by Contradiction. Assume that $S = \{n \mid P(n) \text{ is false}\}$ is non-empty. Let the least element of S be m. Observe that $n_0 \notin S$, therefore $m > n_0$. Furthermore, by minimality, $m - 1 \notin S$. However by the inductive step, $P(m - 1)$ implies $P(m)$, contradiction. □

Using induction, we have a more rigorous way to prove algebraic identities.

Example 1.6. Prove the sum of cubes identity

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Proof by Induction. When $n = 1$, $1 = 1^2$. We assume the formula is true for $n = k$, hence

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2. \tag{Hypothesis}$$

Adding the next cube to both sides of our assumption gives:

$$\begin{aligned}\left[1^3+2^3+3^3+\cdots+k^3\right]+(k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3\\ &= (k+1)^2\left(\frac{k^2}{4}+k+1\right)\\ &= \left[\frac{(k+1)(k+2)}{2}\right]^2.\end{aligned}$$

This is the sum of cubes formula for $n = k+1$, hence the identity holds for all positive integers. □

Example 1.7. (USAMO) Prove that for every positive integer n, there exists an n-digit number divisible by 5^n all of whose digits are odd.

Solution. We use induction. We begin by showing the first 4 base cases: $5, 75, 375, 9375$.

Assume that the k digit number $N = a_1a_2a_3\cdots a_k$ is divisible by 5^k, hence $N = 5^kA$. We show that for one value of $i \in \{1,3,5,7,9\}$ the number below is divisible by 5^{k+1}:

$$N_i = ia_1a_2a_3\cdots a_k = i\cdot 10^k + 5^kA = 5^k\left(i\cdot 2^k + A\right).$$

Since $\{1,3,5,7,9\}$ all give different remainders when divided by 5, there exists an odd i such that 5 divides $i\cdot 2^k + A$. We will define this more in Chapter 2. For this value of i, N_i is a $k+1$-digit number divisible by 5^{k+1} all of whose digits are odd. □

Principle (Strong Mathematical Induction)**.** To prove a statement P for all positive integers $\geq n_0$,

(1) **Base Case:** Show $P(n_0)$.

(2) **Inductive Step:** Show $P(n_0), P(n_0+1), \cdots, P(k)$ implies $P(k+1)$ for any integer $k \geq n_0$.

The most common example of strong induction is the proof of the Fundamental Theorem of Arithmetic, which we will see in Chapter 3. For now, we show several challenge problems.

Example 1.8 (Putnam)**.** Prove that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another.

Solution. We use strong induction. For a base case, 1 is obvious. Assume every integer up to n can be written in this form. We then show that n can also be by breaking it into two cases:

- If n is even, then $n/2$ can be written as a sum by hypothesis. Multiplying every term in this sum by 2 gives the desired representation for n. For example, $5 = 2+3$ and $10 = 4+6$.
- If n is odd, then find s such that $3^s \leq n < 3^{s+1}$. Clearly if $3^s = n$, then we are done. If $3^s < n$, then let $n' = (n-3^s)/2$. Since n' is an integer, it can be written as a sum. Notice the powers of 3 in the representation of n' are less than 3^s since

$$n' = \frac{n-3^s}{2} < \frac{3^{s+1}-3^s}{2} = 3^s.$$

  Multiplying the representation of n' by 2 gives one for $2n'$. We know none of the terms of this sum are divisible by 3^s. Also since they are all even, none divide 3^s. Putting together the representations for $2n'$ with 3^s gives a valid representation for n.

□

Example 1.9 (USAMO)**.** We call an integer n *good* if we can write $n = a_1 + a_2 + \cdots + a_k$, where $a_1, a_2, \cdots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

Solution. We use induction. Observe that if n is good, then

$$\frac{1}{2a_1} + \cdots + \frac{1}{2a_k} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \implies 2n + 8 \text{ is good.}$$
$$\frac{1}{2a_1} + \cdots + \frac{1}{2a_k} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1 \implies 2n + 9 \text{ is good.}$$

Let $P(n)$ be the proposition "all the integers $n, n+1, n+2, \cdots, 2n+7$ are good". The base case $P(33)$ is given. If k is good, then $2k+8$ and $2k+9$ are also good, hence $P(k) \implies P(k+1)$. □

Exercises

1.2.1. Prove that the sum of the first n positive odd integers is n^2.

1.2.2. Prove the geometric series formula for all positive integers n,

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

1.2.3. Prove the Principle of Strong Induction from the well-ordering principle

§1.3 Binomial Coefficients

Induction allows us to define recursive sequences whose terms depend upon previous ones.

Definition. The factorial of a positive n is recursively defined by $n! = n \cdot (n-1)!$ and $0! = 1$. In other words, it equals the product of all positive integers less than or equal to n.

For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

Example 1.10. Prove $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$ for all positive integers n.

Solution. When $n = 1$, $1 = 2! - 1$. We now assume the identity holds for $n = k$, therefore

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1 \qquad \text{(Hypothesis)}$$

We add the next factorial to both sides of our hypothesis:

$$\begin{aligned}[1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!] + (k+1) \cdot (k+1)! &= [(k+1)! - 1] + (k+1) \cdot (k+1)! \\ &= (k+2) \cdot (k+1)! - 1 \\ &= (k+2)! - 1.\end{aligned}$$

Therefore, the identity holds for all positive integers by induction. □

Definition. We also define the binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. For example, $\binom{4}{2} = 6$.

Intuitively, a binomial coefficient is the number of ways to choose k people out of n to be on a committee if we do not care about the order in which we select the people

Theorem 1.1 (Pascal's Identity).

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. By the definition of binomial coefficients and factorials,

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= (n-1)!\left(\frac{k}{k!(n-k)!} + \frac{n-k}{k!(n-k)!}\right) \\ &= (n-1)!\left(\frac{n}{k!(n-k)!}\right) \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}.\end{aligned}$$

□

Using this identity, we can create a triangle out of all the binomial coefficients, which is known as **Pascal's triangle**. We leave a proper treatment of Pascal's triangle to a book on combinatorics, but we do explore some interesting identities now.

Example 1.11. In this problem, we will prove $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(i) Prove that $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

(ii) Prove that $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$ for $n \geq 2$.

(iii) From part (i) and (ii), deduce the sum of squares formula.

Solution. (i) When $n = 1$, $1 = 1$. We now assume the identity holds for $n = k$, therefore

$$1 + 2 + 3 + \cdots + k = \binom{k+1}{2}. \qquad \text{(Hypothesis)}$$

Adding the next integer to both sides of our hypothesis:

$$\begin{aligned}[1 + 2 + 3 + \cdots + k] + k + 1 &= \binom{k+1}{2} + k + 1 \\ &= \binom{k+2}{2}.\end{aligned}$$

The last step follows from either Pascal's identity or simple algebraic manipulation.

(ii) When $n = 2$, $\binom{2}{2} = \binom{3}{3}$. We now assume the identity holds for $n = k$, therefore

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k}{2} = \binom{k+1}{3}. \qquad \text{(Hypothesis)}$$

Adding the next binomial to both sides of our hypothesis:

$$\begin{aligned}\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k}{2}\right] + \binom{k+1}{2} &= \binom{k+1}{3} + \binom{k+1}{2} \\ &= \binom{k+2}{3}. \qquad \text{(Pascal's Identity)}\end{aligned}$$

Therefore, the identity holds for all $n \geq 2$ by induction.

(iii) Observe that $k^2 = 2\binom{k}{2} + k$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} \left[2\binom{k}{2} + k\right] \\ &= 2\binom{n+1}{3} + \binom{n+1}{2} \\ &= 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

□

We have now seen formulas for the sum of squares and sum of cubes. You will be asked, as a fun bonus challenge, to derive a formula for the sum of fourth powers.

Using Pascal's identity along with induction, we can prove the following result:

Theorem 1.2 (Binomial Theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof. We begin by presenting a formal proof with induction, and then present an intuitive proof.

When $n = 1$, $(x+y)^1 = x + y$. Assuming the binomial theorem for $n - 1$:

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k-1} y^k. \qquad \text{(Hypothesis)}$$

Multiplying by $x+y$, we see that

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\left[\sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k-1}y^k\right] \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k-1}y^{k+1} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{k'=1}^{n}\binom{n-1}{k'-1}x^{n-k'}y^{k'} && (k'=k+1) \\
&= x^n + \sum_{k=1}^{n-1}\left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]x^{n-k}y^k + y^n \\
&= x^n + \sum_{k=1}^{n-1}\binom{n}{k}x^{n-k}y^k + y^n && \text{(Pascal's Identity)} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.
\end{aligned}$$

Therefore, the binomial theorem holds for all positive integers n by induction. □

Intuitively, the binomial theorem is true since in the expansion of

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}},$$

to get the coefficient of $x^{n-k}y^k$, we have to choose k out of the n factors to be y. By the definition of binomial coefficients, the number of ways to accomplish this is $\binom{n}{k}$, hence this is the coefficient.

Example 1.12. Prove that if n is a positive integer, then $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}$ is always an integer.

Solution. When $n=1$, $1/5+1/2+1/3-1/30=1$. Assume that $k^5/5+k^4/2+k^3/3-k/30$ is an integer for an arbitrary k. We expand the expression for $n=k+1$ using the Binomial Theorem:

$$\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^4+4k^3+6k^2+4k+1}{2}-\frac{k^3+3k^2+3k+1}{3}-\frac{k+1}{30}.$$

With some algebraic manipulation, this expression is equivalent to

$$\left(\frac{k^5}{5}+\frac{k^4}{2}+\frac{k^3}{3}-\frac{k}{30}\right)+\left(k^4+2k^3+2k^2+k+2k^3+3k^2+2k+k^2+k+1\right),$$

which is an integer by the induction hypothesis. □

Exercises

1.3.1. Prove that $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ if $n \geq k \geq r \geq 0$.

1.3.2. Prove that $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$.

1.3.3. Prove the following identies using the Binomial Theorem:

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

(ii) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n} = 0$.

(iii) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$.

(iv) $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n} = 3^n$.

§1.4 Fibonacci Numbers

Definition. The Fibonacci numbers are defined by $F_1 = 1, F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Every number is the sum of the two preceeding terms. The first several Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots .$$

The Fibonacci numbers have many beautiful and surprising properties.

Example 1.13. Consider a board of length n. How many ways are there to tile this board with squares (length 1) and dominoes (length 2)?

Solution. Let $f(n)$ be the number of tilings of an n-board. We can compute $f(1) = 1$ and $f(2) = 2$. Depending on if we begin with a square or domino, we either have a $n-1$ or a $n-2$ board remaining:

$$f(n) = f(n-1) + f(n-2).$$

This is exactly the Fibonacci recurrence with a shifted index of 1. Hence, $f(n) = F_{n+1}$. □

Example 1.14. Prove that for all positive integers n,

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

Proof by Induction. When $n = 1$, $1 = 2 - 1$. Assuming the identity for $n = k$, we have

$$F_1 + F_2 + F_3 + \cdots + F_k = F_{k+2} - 1. \qquad \text{(Hypothesis)}$$

We add the next Fibonacci number, F_{k+1}, to both sides of our assumption (in parenthesis):

$$\begin{aligned}[F_1 + F_2 + F_3 + \cdots + F_k] + F_{k+1} &= [F_{k+2} - 1] + F_{k+1} \\ &= F_{k+3} - 1.\end{aligned}$$

This is the identity for $n = k + 1$, hence by induction, our proof is complete. □

Proof by Counting. Consider the number of tilings of a board of size $n+1$. Of these, $f(n+1)-1$ use at least one domino. We consider the position of the final domino, that is, the location $1 \leq k \leq n$ such that there is a domino covering cells k and $k + 1$ and all squares beyond that point. We therefore simply have to tile the first $k - 1$ squares of the board, which can be done in $f(k - 1)$ ways. Therefore, $F_{n+2} - 1 = f(n + 1) - 1 = \sum_{k=1}^{n} f(k - 1) = \sum_{k=1}^{n} F_k = F_1 + \cdots + F_n$. □

Example 1.15. Prove that the diagonal sum of Pascal's triangle are Fibonacci numbers,

$$F_{n+1} = \sum_{k \geq 0} \binom{n-k}{k} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots .$$

Solution. The left-hand side is the number of tilings of an n-board. If there are k dominoes in a tiling, then there are $n - 2k$ squares for a total of $n - k$ tiles. The number of ways to select k of these to be dominoes is $\binom{n-k}{k}$. Therefore, there are $f(n) = F_{n+1} = \sum_{k \geq 0} \binom{n-k}{k}$ tilings. □

Example 1.16. Prove that $F_{s+t} = F_{s+1}F_t + F_sF_{t-1}$ for integers $s \geq 0$ and $t \geq 1$.

Solution. Shifting the indices, we desire to prove $f(s+t) = f(s)f(t) + f(s-1)f(t-1)$. The LHS is the number of tilings of an $(s+t)$-board. We condition on if there is a domino at s in our tiling:

(i) If there is no domino at s, we have $f(s)f(t)$ tilings of the $(s + t)$-board.

(ii) If there is a domino at s, we have $f(s - 1)f(t - 1)$ tilings of the $(s + t)$-board.

Therefore, we have established that $f(s + t) = f(s)f(t) + f(s - 1)f(t - 1)$. □

Example 1.17 (Binet's Formula)**.** Recall $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\psi = \frac{1-\sqrt{5}}{2}$. Prove

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

Proof by Induction. For base cases, $F_1 = (\varphi - \psi)/\sqrt{5} = 1$ and $F_2 = (\varphi^2 - \psi^2)/\sqrt{5} = 1$. Furthermore, φ and ϕ are roots of the quadratic $x^2 - x - 1 = 0$, therefore

$$\begin{aligned}\varphi^k &= \varphi^{k-1} + \varphi^{k-2} \\ \psi^k &= \psi^{k-1} + \psi^{k-2}.\end{aligned}$$

Assume Binet's formula for $n = k - 2$ and $n = k - 1$. From the definition of Fibonacci numbers,

$$\begin{aligned} F_k &= F_{k-1} + F_{k-2} \\ &= \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} + \frac{\varphi^{k-2} - \psi^{k-2}}{\sqrt{5}} \\ &= \frac{\varphi^{k-1} + \varphi^{k-2}}{\sqrt{5}} - \frac{\psi^{k-1} + \psi^{k-2}}{\sqrt{5}} \\ &= \frac{\varphi^k - \psi^k}{\sqrt{5}}. \end{aligned}$$

This is Binet's formula for k, hence we have proven the identity by induction. □

1.4.1 Exercises

1.4.1. Prove the following Fibonacci identities in two different ways:

(i) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

(ii) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

1.4.2. (Cassini's Identity) Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

§1.5 Recursion

A recursive function is a function that is defined in terms of itself. We have already seen several examples of recursive functions, including the factorial function and the Fibonacci numbers. In this section, we'll examine a few more recursive functions.

Example 1.18. The Ackermann function is a recursive function defined by

$$A(m,n) = \begin{cases} n+1, & \text{if } m = 0 \\ A(m-1,1), & \text{if } n = 0 \\ A(m-1, A(m, n-1)) & \text{otherwise.} \end{cases}$$

Prove that for every natural n, (i) $A(1,n) = n+2$, (ii) $A(2,n) = 2n+3$, (iii) $A(3,n) = 2^{n+3} - 3$.

Proof by Induction. We use these three examples to build up the Ackermann function.

(i) When $n = 0$, $A(1,0) = A(0,1) = 2$. If $A(1,k) = k+2$ for a positive integer k. Then,

$$A(1,k+1) = A(0, A(1,k)) = A(0, k+2) = k+2+1 = k+3.$$

(ii) When $n = 0$, $A(2,0) = A(1,1) = 3$. Assume $A(2,k) = 2k+3$ for a positive integer k. Then,

$$A(2,k+1) = A(1, A(2,k)) = A(1, 2k+3) = (2k+3)+2 = 2(k+1)+3.$$

(iii) When $n = 0$, $A(3, 0) = A(2, 1) = 5$. Assume $A(3, k) = 2^{k+3} - 3$ for $k \in \mathbb{Z}$. Then,

$$A(3, k+1) = A(2, A(2, k)) = A(2, 2^{k+3} - 3) = 2 \cdot \left(2^{k+3} - 3\right) + 3 = 2^{k+4} - 3. \qquad \square$$

In computability theory, the Ackermann function was the earliest-discovered total computable function that is not primitive recursive, meaning it can't be rewritten using for loops. It is often used as a benchmark of a compiler's ability to optimize deep recursion. To compute larger values of the function, we introduce notation first discovered by Donald Knuth in 1976.

Definition. Knuth's up-arrow notation is a method of notation for very large integers.

- Single arrow is exponentiation: $a \uparrow n = a^n$.
- Double arrow is iterated exponentiation, known as tetration:

$$a \uparrow\uparrow n = \underbrace{a \uparrow (a \uparrow (a \uparrow (\cdots a \uparrow a))}_{n\ a's}.$$

  For example, $2 \uparrow\uparrow 4 = 2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow 2))) = 2^{2^{2^2}} = 65536$.

- Triple arrow is iterated tetration:

$$a \uparrow\uparrow\uparrow n = \underbrace{a \uparrow\uparrow (a \uparrow\uparrow (a \uparrow\uparrow (\cdots a \uparrow\uparrow a)))}_{n\ a's}.$$

- In general, we define the up-arrow notation recursively as

$$a \uparrow^n b = \begin{cases} 1 & \text{if } n \geq 1 \text{ and } b = 0 \\ a \uparrow^{n-1} (a \uparrow^n (b-1)) & \text{otherwise.} \end{cases}$$

For $m = 4$, $A(4, n) = 2 \uparrow\uparrow^{m-n} (n+3) - 3$. For example, $A(4, 0) = 13$, $A(4, 1) = 65533$, and

$$A(4, 2) = 2^{2^{2^{2^2}}} - 3 = 2^{65536} - 3.$$

This has 19729 decimal digits! In general, we can prove $A(m, n) = 2 \uparrow^{m-2} (n+3) - 3$.

Definition. Graham's number is the enormous number g_{64} in the recursive definition

$$g_n = \begin{cases} 3 \uparrow\uparrow\uparrow\uparrow 3, & n = 1 \\ 3 \uparrow^{g_{n-1}} 3, & n \geq 2. \end{cases}$$

Notice the number of arrows in each subsequent layer is the value of the layer proceeding it.

To begin to understand the depth of Graham's number, we show the first several power towers:

$$3 = 3, \quad 3^3 = 27, \quad 3^{3^3} = 7,625,597,484,987.$$

We define the *sun tower* as $3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow 3^{3^3}$, a power tower with 7.6 trillion 3's. Then, $g_1 = 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3)$ is the result of applying the function $x \mapsto 3 \uparrow\uparrow x$ a sun tower amount of times beginning with $x = 1$. Finally, Graham's number is a stacked up-arrow tower:

$$G = \left.\begin{matrix} 3\underbrace{\uparrow\uparrow\cdots\cdots\cdots\cdots\cdots\uparrow}3 \\ 3\underbrace{\uparrow\uparrow\cdots\cdots\cdots\cdots\uparrow}3 \\ \underbrace{\qquad\vdots\qquad} \\ 3\underbrace{\uparrow\uparrow\cdots\cdots\uparrow}3 \\ 3\uparrow\uparrow\uparrow\uparrow 3 \end{matrix}\right\}\text{64 layers}$$

Exercises

1.5.1. The McCarthy 91 function is a recursive function defined by

$$M(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ M(M(n+11)), & \text{if } n \le 100. \end{cases}$$

Prove that $M(n) = 91$ for all integers $n \le 100$.

1.5.2. Define a_n by $a_0 = 2$, $a_1 = 8$, and $a_n = 8a_{n-1} - 15a_{n-2}$ for $n \ge 2$. Prove that $a_n = 3^n + 5^n$.

§1.6 Challenge Problems

1.19. (i) For real x, define $x_n = x^n + \frac{1}{x^n}$. Find x_2, x_3, x_4, and x_5 in terms of x_1.

(ii) Prove that if x_1 is an integer, then x_n is always an integer for all natural n.

1.20. Prove that $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for all positive integers n.

1.21⋆ Evaluate the telescoping sum

$$\sum_{k=1}^{n} \left[(k+1)^5 - k^5\right]$$

in two separate ways to find a formula for the sum of fourth powers.

1.22⋆ Evaluate the sum

$$\binom{2025}{0} + \binom{2025}{3} + \binom{2025}{6} + \cdots + \binom{2025}{2025}.$$

1.23⋆ (Zeckendorf) Prove every positive integer N can be represented as the unique sum of non-consecutive Fibonacci numbers. In other words, there exists a unique $\{a_j\}_{j=0}^{m}$ with

$$N = \sum_{j=0}^{m} F_{a_j}, \quad a_0 \ge 2 \text{ and } a_{j+1} > a_j + 1.$$

APPENDIX A

SOLUTIONS

§A.1 The Art of Proofs Solutions

Exercises for Section 1.2

1.2.1 When $n = 1$, $1 = 1^2$. Assume the identity holds for $n = k$, therefore

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2. \qquad \text{(Hypothesis)}$$

We add the next odd number to both sides of the hypothesis:

$$\begin{aligned}[1 + 3 + 5 + \cdots + (2k - 1)] + 2k + 1 &= \left[k^2\right] + 2k + 1 \\ &= (k + 1)^2 .\end{aligned}$$

Therefore, the identity holds for all positive integers n by induction.

1.2.2 When $n = 1$, $1 = 1$. Assume the geometric series holds for $n = k$, therefore

$$1 + r + r^2 + \cdots + r^{k-1} = \frac{r^k - 1}{r - 1}. \qquad \text{(Hypothesis)}$$

We add the next term to both sides of the hypothesis:

$$\begin{aligned}\left[1 + r + r^2 + \cdots + r^{k-1}\right] + r^k &= \left[\frac{r^k - 1}{r - 1}\right] + r^k \\ &= \frac{r^k - 1 + r^{k+1} - r^k}{r - 1} \\ &= \frac{r^{k+1} - 1}{r - 1}.\end{aligned}$$

Therefore, the geometric series formula holds for all positive integers n by induction.

1.2.3 Assume that $S = \{n \mid P(n) \text{ is false}\}$ is non-empty. Let the least element of S be m. Observe that $n_0 \notin S$, therefore $m > n_0$. Furthermore, since m is the smallest element of S, $P(n)$ is true for all $n_0 \leq n \leq m - 1$. However, by the inductive step, this implies $P(m)$ is also true, contradiction.

Exercises for Section 1.3

1.3.1 We see that the left hand side is

$$\binom{n}{k}\binom{k}{r} = \frac{n!}{k!(n-k)!} \cdot \frac{k!}{r!(k-r)!} = \frac{n!}{(n-k)!r!(k-r)!}.$$

Similarly, the right hand side is

$$\binom{n}{r}\binom{n-r}{k-r} = \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!(n-k)!} = \frac{n!}{(n-k)!r!(k-r)!}.$$

Therefore, the identity is proven.

1.3.2 Each term in our sum is equivalent to

$$\begin{aligned} k\binom{n}{k} &= k\left(\frac{n!}{k!(n-k)!}\right) \\ &= \frac{n!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}. \end{aligned}$$

Alternatively, we can see this by setting $r = 1$ in the previous problem. Therefore, we can rewrite the summation as

$$\begin{aligned} \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} &= n\left[\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}\right] \\ &= n2^{n-1}. \end{aligned}$$

1.3.3 (i) When $x = y = 1$ in the Binomial Theorem, we have

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(ii) When $x = 1$ and $y = -1$ in the Binomial Theorem, we have

$$(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

(iii) Adding the two previous identities gives us

$$2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right] = 2^n.$$

Dividing by 2 gives us that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$.

(iv) When $x = 1$ and $y = 2$ in the Binomial Theorem, we have

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots = 3^n.$$

Exercises for Section 1.4

1.4.1 (i) When $n = 1$, $F_1 = F_2$. Assume the identity holds for $n = k$, therefore

$$F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}. \qquad \text{(Hypothesis)}$$

Adding the next odd Fibonacci term to our hypothesis:

$$\begin{aligned}[F_1 + F_3 + F_5 + \cdots + F_{2k-1}] + F_{2k+1} &= [F_{2k}] + F_{2k+1} \\ &= F_{2k+2}.\end{aligned}$$

Therefore, the Fibonacci identity holds for all positive integers n by induction.

Alternatively, for a combinatorial proof, consider tiling a board of length $2n-1$, which can be done in $f_{2n-1} = F_{2n}$ ways. Condition on the location of the first square on the board. If the first square is at the first location, then the remaining $2n-2$ squares can be tiled in $f_{2n-2} = F_{2n-1}$.

Notice the first square cannot be at the second location, so the next case is the first square is at the third location, which means we start with one domino. In this case, we need to tile the remaining $(2n-1) - 3 = 2n - 4$ squares, which can be done in $f_{2n-4} = F_{2n-3}$ ways. Continuing on the casework in this way, we observe that we get:

$$f_0 + f_2 + \cdots + f_{2n-4} + f_{2n-2} = f_{2n-1} \implies F_1 + F_3 + F_5 + \cdots + F_{2n-3} + F_{2n-1} = F_{2n}.$$

(ii) When $n = 1$, $F_1^2 = F_1 F_2$. Assume the identity holds for $n = k$, therefore

$$F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}. \qquad \text{(Hypothsis)}$$

Adding the square of the next Fibonacci number to our hypothesis:

$$\begin{aligned}\left[F_1^2 + F_2^2 + \cdots + F_k^2\right] + F_{k+1}^2 &= [F_k F_{k+1}] + F_{k+1}^2 \\ &= F_{k+1}\left(F_k + F_{k+1}\right) \\ &= F_{k+1} F_{k+2}.\end{aligned}$$

Therefore, the Fibonacci identity holds for all positive integers n by induction.

For an algebraic proof, by the definition of Fibonacci numbers we know that

$$F_{n+2} = F_{n+1} + F_n.$$

Multiplying this by F_{n+1} gives us:

$$F_{n+2} F_{n+1} = F_{n+1}^2 + F_{n+1} F_n.$$

Summing up this equation from $n = 0$ to $n = k - 1$ gives us:

$$\sum_{n=0}^{k-1} F_{n+2} F_{n+1} = \sum_{n=0}^{k-1} F_{n+1}^2 + \sum_{n=0}^{k-1} F_{n+1} F_n.$$

Now observe that all the terms in the first and third term cancel except for $F_{k+1} F_k$ and $F_1 F_0 = 0$, so we're left with:

$$F_{k+1} F_k = \sum_{n=0}^{k-1} F_{n+1}^2 = F_1^2 + F_2^2 + \cdots + F_k^2.$$

1.4.2 Notice $\varphi\psi = -1$ and $\varphi - \psi = \sqrt{5}$. Using Binet's formula and algebraic manipulation,

$$\begin{aligned} F_{n+1}F_{n-1} - F_n^2 &= \frac{1}{5}\left[\left(\varphi^{n+1} - \psi^{n+1}\right)\left(\varphi^{n-1} - \psi^{n-1}\right) - (\varphi^n - \psi^n)^2\right] \\ &= \frac{1}{5}\left[-\varphi^{n+1}\psi^{n-1} - \psi^{n+1}\varphi^{n-1} + 2\varphi^n\psi^n\right] \\ &= -\frac{1}{5}(\varphi\psi)^{n-1}\left(\varphi^2 - 2\varphi\psi + \psi^2\right) \\ &= -\frac{1}{5}(-1)^{n-1}(\varphi - \psi)^2 \\ &= (-1)^n. \end{aligned}$$

Exercise for Section 1.5

1.5.1 We use strong induction. For a base case, if $90 \le k < 101$, then $k + 11 > 100$, so

$$M(k) = M(M(k+11)) = M(k + 11 - 10) = M(k+1).$$

Therefore, $M(90) = M(91) = \cdots = M(100) = M(101) = 101 - 10 = 91$. We now use induction on blocks of 11 numbers. Assume that $M(k) = 91$ for $a \le k < a + 11$. Then, for $a - 11 \le k < a$,

$$M(k) = M(M(k+11)) = M(91) = 91.$$

Since we established the base case $a = 90$, $M(k) = 91$ for any k in such a block. Letting a be multiples of 10, there are no holes between the blocks, hence $M(k) = 91$ for all integers $k \le 100$.

1.5.2 We see $a_0 = 3^0 + 5^0 = 2$ and $a_1 = 3^1 + 5^1 = 8$. Assume the formula holds for $n = k - 2$ and $n = k - 1$. We then show it holds for $n = k$. Note that 3 and 5 are roots of $x^2 - 8x + 15 = 0$, hence

$$3^k = 8 \cdot 3^{k-1} - 15 \cdot 3^{k-2}, \quad 5^k = 8 \cdot 5^{k-1} - 15 \cdot 5^{k-2}.$$

Using these identities along with the inductive hypothesis,

$$\begin{aligned} a_k &= 8a_{k-1} - 15a_{k-2} \\ &= 8\left(3^{k-1} + 5^{k-1}\right) - 15\left(3^{k-2} + 5^{k-2}\right) \\ &= 3^k + 5^k. \end{aligned}$$

Review Problems

1.19 (i) Squaring x_1, we see $x_1^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$, therefore $x_2 = \boldsymbol{x_1^2 - 2}$. Cubing x_1, we see

$$x_1^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} \implies x_3 = \boldsymbol{x_1^3 - 3x_1}.$$

Squaring the equation for x_2 gives an expression for x_4:

$$x_2^2 = \left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + 2 + \frac{1}{x^4} \implies x_4 = x_2^2 - 2 = \boldsymbol{x_1^4 - 4x_1^2 + 2}.$$

Finally, to find x_5, we multiply x_1 by x_4 to get a recursive relation,

$$x_1x_4 = \left(x + \frac{1}{x}\right)\left(x^4 + \frac{1}{x^4}\right) = x^5 + \frac{1}{x^5} + x^3 + \frac{1}{x^3} \implies x_5 = x_1x_4 - x_3 = \boldsymbol{x_1^5 - 5x_1^3 + 5x_1}.$$

(ii) Since x_1 is an integer, x_2, x_3, x_4, and x_5 are all integers. Inspired by our work for x_5,

$$x_1 a_{k-1} = \left(x + \frac{1}{x}\right)\left(x^{k-1} + \frac{1}{x^{k-1}}\right) = x^k + x^{k-2} + \frac{1}{x^{k-2}} + \frac{1}{x^k}.$$

Assume that x_{k-1} and x_{k-2} are both integers. Then, $x_k = x_1 a_{k-1} - a_{k-2}$ is also an integer. By induction, since we showed several base cases, x_n is an integer for all positive integers n.

1.20 When $n = 1$, $1/2 = 1/2$. We now assume the identity holds for $n = k$, therefore

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}. \qquad \text{(Hypothesis)}$$

We add the next fraction to both sides of our assumption:

$$\begin{aligned}\left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}.\end{aligned}$$

Therefore, the identity holds for all positive integers by induction. Alternatively, we could have solved this problem by noticing that the sum telescopes:

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)}.$$

1.21 Using the binomial theorem as well as the formulas we previously derived in this chapter:

$$\begin{aligned}\sum_{k=1}^{n}\left[(k+1)^5 - k^5\right] &= (n+1)^5 - 1\\ \sum_{k=1}^{n}\left(5k^4 + 10k^3 + 10k^2 + 5k + 1\right) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n\\ 5\sum_{k=1}^{n}k^4 + 10\frac{n^2(n+1)^2}{4} + 10\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} + n &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n\end{aligned}$$

We subtract n from the right-hand side and factor out $n(n+1)$ using synthetic division to get:

$$\begin{aligned}5\sum_{k=1}^{n}k^4 + \frac{10n(n+1)}{4}n(n+1) + \frac{10(2n+1)}{6}n(n+1) + \frac{5}{2}n(n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 4n\\ &= n(n+1)\left(n^3 + 4n^2 + 6n + 4\right).\end{aligned}$$

Subtracting all the like terms and doing some algebraic simplifications gives us:

$$5\sum_{k=1}^{n}k^4 = n(n+1)\left(n^3 + \frac{3}{2}n^2 + \frac{1}{6}n - \frac{1}{6}\right) = n(n+1)\frac{6n^3+9n^2+n-1}{6}.$$

Observe that $n = -\frac{1}{2}$ is a root of the cubic in the numerator; therefore, using synthetic division, we find $6n^3 + 9n^2 + n - 1 = (2n+1)(3n^2+3n-1)$. Substituting this back into our formula gives us:

$$5\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{6}.$$

Finally, dividing by 5 gives the formula

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

1.22 Let ω be the third root of unity, that is, it is a complex number that is not 1 that satisfies $\omega^3 = 1$ or equivalently $\omega^2 + \omega + 1 = 0$. Now, using the binomial theorem, we expand the following terms to get:

$$\begin{aligned}(1+\omega)^{2025} &= \binom{2025}{0} + \binom{2025}{1}\omega + \binom{2025}{2}\omega^2 + \binom{2025}{3}\omega^3 + \cdots \\ (1+\omega^2)^{2025} &= \binom{2025}{0} + \binom{2025}{1}\omega^2 + \binom{2025}{2}\omega^4 + \binom{2025}{3}\omega^6 + \cdots \\ (1+1)^{2025} &= \binom{2025}{0} + \binom{2025}{1}1 + \binom{2025}{2}1 + \binom{2025}{3} + \cdots\end{aligned}$$

Summing up all the terms with $\binom{2025}{1}$ gives us a coefficient of $\omega + \omega^2 + 1 = 0$. Similarly, summing up all the terms with $\binom{2025}{2}$ gives a coefficient of $\omega^2 + \omega^4 + 1 = \omega^2 + \omega + 1 = 0$. Notice that since $\omega^3 = 1$, the pattern will repeat every three terms, and only the terms with a multiple of 3 in the bottom part of the binomial coefficient will not cancel. Therefore, we have:

$$(1+\omega)^{2025} + (1+\omega^2)^{2025} + (1+1)^{2025} = 3\left[\binom{2025}{0} + \binom{2025}{3} + \cdots + \binom{2025}{2025}\right].$$

Notice that $1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$. Also since 2025 is a multiple of 3, we have $\omega^{2025} = 1$, so:

$$\begin{aligned}3\left[\binom{2025}{0} + \binom{2025}{3} + \cdots + \binom{2025}{2025}\right] &= (-\omega^2)^{2025} + (-\omega)^{2025} + (2)^{2025} \\ &= -1 - 1 + 2^{2025}\end{aligned}$$

Hence, our desired sum is

$$\binom{2025}{0} + \binom{2025}{3} + \cdots + \binom{2025}{2025} = \frac{2^{2025} - 2}{3}.$$

1.23 For the base case of $N = 1$, the unique representation sum is $1 = F_2$. Now, assume that every every integer up to K can be written as the unique sum of distinct non-consecutive Fibonacci numbers. Let $F_{\max}$ be the largest Fibonacci number such that $F_{\max} \le K + 1$. If $F_{\max} = K + 1$, then we are clearly done. Otherwise, $F_{\max} < K + 1 < F_{\max+1}$, therefore

$$0 < (K+1) - F_{\max} < F_{\max+1} - F_{\max} = F_{\max-1}. \qquad (\star)$$

By our hypothesis, there exists a sequence $\{a_j\}_{j=0}^{m}$ with $a_{j+1} > a_j + 1$ such that

$$K + 1 - F_{\max} = \sum_{j=0}^{m} F_{a_j}.$$

Since $F_{a_m} < F_{\max -1}$ by $(\star)$, adding $F_{\max}$ to both sides produces a valid representation for $K+1$.

For uniqueness, we require the following lemma, whose proof is left as an exercise:

Lemma. *The sum of any set of distinct, non-consecutive Fibonacci numbers whose largest member is F_j is strictly less than the next larger Fibonacci number F_{j+1}.*

Proof. We prove this using strong induction. We begin by proving that this is true for $j = 2$. In this case, we see that $F_2 = 1 < F_3 = 2$. Assume this is true for all values up to $j = k$. We then prove this is true for $j = k+1$. To prove this, observe that the set of distinct, non-consecutive Fibonacci numbers for $j = k+1$ can be at most $F_{k+1} + F_{k-1} + \cdots$. The sum of all the elements after F_{k-1} is the sum of a set of distinct, non-consecutive Fibonacci numbers whose largest member is F_{k-1}. Therefore, by the induction hypothesis, this sum is at most F_k:

$$F_{k+1} + F_{k-1} + \cdots < F_{k+1} + F_k = F_{k+2}.$$

Therefore, this sum is less than the next Fibonacci number, and we've proven this result is true using strong induction. □

For the sake of contradiction, let $K+1$ be the smallest integer with two representations:

$$\begin{aligned} K + 1 &= F_{a_1} + F_{a_2} + \cdots + F_{a_m} \\ &= F_{b_1} + F_{b_2} + \cdots + F_{b_l}. \end{aligned}$$

Without loss of generality, assume that $a_m \geq b_l$. If $a_m > b_l$, then our Lemma shows

$$\begin{aligned} K + 1 &= F_{b_1} + F_{b_2} + \cdots + F_{b_l} \\ &< F_{b_l+1} - 1 \\ &\leq F_{a_m} - 1 \\ &< F_{a_1} + F_{a_2} + \cdots + F_{a_m} \\ &= K + 1. \end{aligned}$$

This is a contradiction, therefore $a_m = b_l$. By our hypothesis, $K + 1 - F_{a_m} = K + 1 - F_{b_l}$ has a unique representation, so adding the values back, $K+1$ also has a unique representation. The method of subtracting the largest Fibonacci number is known as a **greedy strategy**.