Surprises from Puzzle Video Games

Undistinguished Lecture Series

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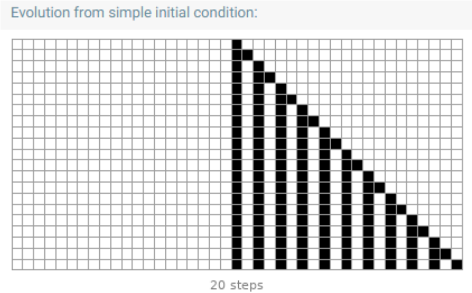
Justin Stevens (jdsteven@ualberta.ca)

Outline

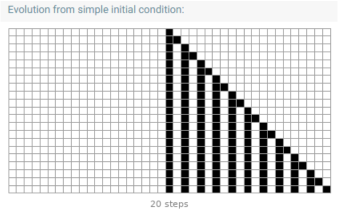
Cellular Automata Example

Consider rule 28 in cellular automatas, which updates a 1 dimensional grid of cells by scanning patterns of three consecutive cells according to the rules:

If we apply the rule for 20 steps, we get the following diagram:



Converting Cellular Automata to Generating Functions



If we consider each individual row of the diagram above, we can convert it into a binary number:

Binary Number	Decimal
1_2	1
11_2	3
101_2	5
1011_2	11
10101_2	21
101011_2	43

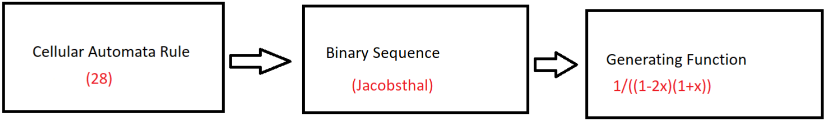
Converting this to a Generating Function

This sequence of numbers $1, 3, 5, 11, 21, 43$ satisfies a simple rule: $a_{n+1} = a_n + 2a_{n-1}$, starting from $a_0 = 1, a_1 = 1$. These are known as the Jacobsthal numbers, and can be turned into a generating function using algebraic manipulations, if we let $J(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\begin{aligned} J(x) &= 1 + x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2}) x^n \\ &= 1 + x + \sum_{n=2}^{\infty} (a_{n-1} x^n) + 2 \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= 1 + x + x \left(a_1 x + a_2 x^2 + \cdots \right) + 2x^2 \left(a_0 + a_1 x + \cdots \right) \\ &= 1 + x + x \left(J(x) - a_0 \right) + 2x^2 J(x). \end{aligned}$$

Therefore $J(x)(1 - x - 2x^2) = 1$, so $J(x) = \frac{1}{1-x-2x^2} = \frac{1}{(1-2x)(1+x)}$.

Observations



- None of the behaviour here is *surprising*, however. We started with a simple rule, it produced a predictable pattern, which can then be summarized with a recursive formula or simple algebraic function.
- Can we make small changes to either the algebraic function or cellular automata to produce behaviour that is surprising?

Wolfram's CA Rules

Wolfram divided the 256 basic CA rules he found into 4 classifications.

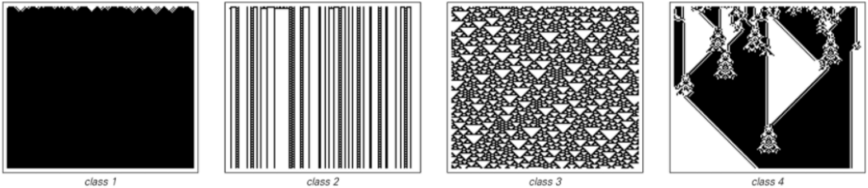
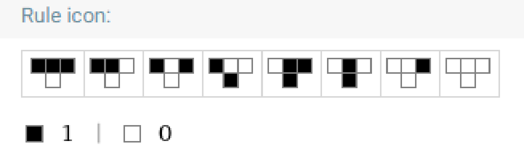


Figure 1: *Source:* Wolfram, A New Kind of Science

- Class 1: Converge to a uniform state quickly.
- Class 2: Converges to a repetitive or stable state, such as rule 28.
- Class 3: Random state, no apparent pattern.
- Class 4: Capable of universal computation.

Producing Class 3 or Class 4 Rules

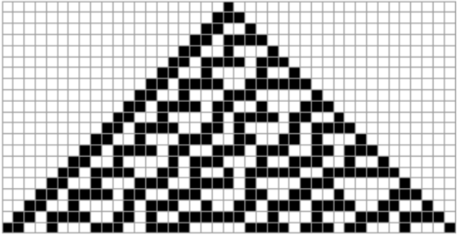
- One might suspect that to produce class 3 or class 4 rules, we need to change the underlying structure of cellular automata a lot.
- However, in reality, we only need to flip a *single* bit from rule 28 to produce a class 3 (random states, no apparent pattern) rule.
- Rule 28:



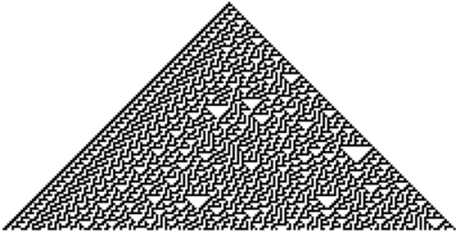
- Rule 30:

Rule 30 Behaviour

Evolution from simple initial condition



20 steps

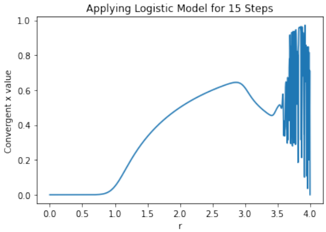


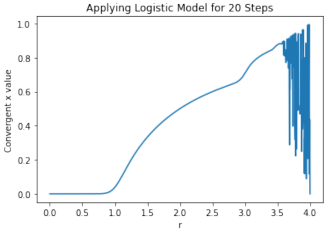
100 steps

Outline

Logistic Map

Consider the rule $x_{n+1} = rx_n(1 - x_n)$, known as the logistic map. Here are the results if we apply this rule for 15 and 20 steps are shown below:





Observations

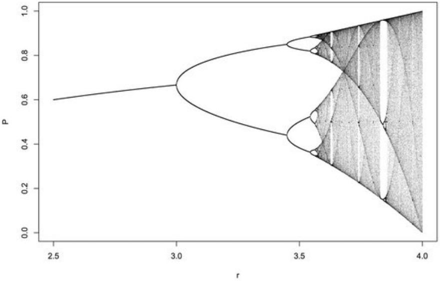


Figure 2: *Source:* Islands of Order

- For $0 \leq r \leq 3$, the update rule converges to the same value, $(r-1)/r$.
- At $r = 3$ we have our first split, depending on whether n is even or odd.
- At $r \approx 3.44949$, there is a second split of values and at $r \approx 3.544$ there is a third split as seen on the diagram.
- After $r \approx 3.56995$, there is chaos in the diagram, since there are no longer patterns for what values the rule converges to.

Feigenbaum Constant

Let the sequence of points when the graph splits be b_i. Considering the ratio $\frac{b_{n-1}-b_{n-2}}{b_n-b_{n-1}}$, this approaches a value known as Feigenbaum's constant.

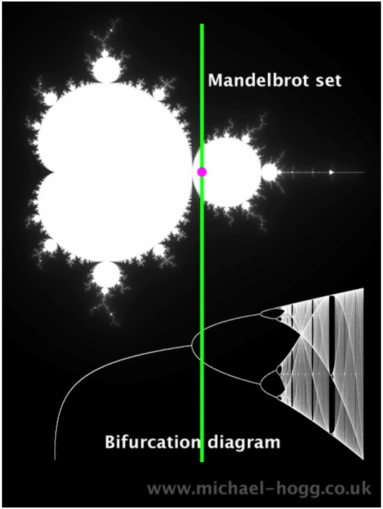
Period	Parameter a	Ratio
2	3.0000000	
4	3.4494896	
8	3.5440903	4.7514
16	3.5644073	4.6562
32	3.5687594	4.6683
64	3.5696916	4.6686
128	3.5698913	4.6692
256	3.5699340	4.6694

Figure 3: *Source:* Chaos: An Introduction to Dynamical Systems by Alligood et al.

The full value of the Feigenbaum constant is $\delta \approx 4.669201609 \cdots$.

Mandelbrot Set and Feigenbaum Constant

The same pattern also occurs in the Mandelbrot set, which is obtained by iteratively applying the function $f(z) = z^2 + c$ from a complex number c.



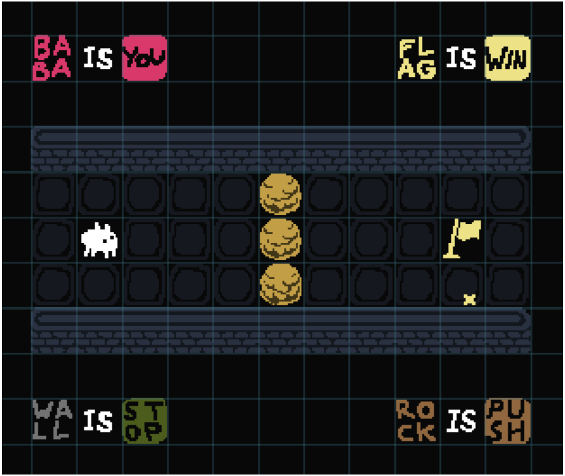
Summary: The Art of Simplicity

- Simple rules lead to dynamic and beautiful emergent behaviour.
- We saw how in the cellular automata, a small change to the dynamics of the rules can lead to a drastic change in the behaviour of the system.
- Similarly, in logistic mapping, a small change in the value of r can create chaos with the various bifurcation points.
- We see applications of the ideas of simplicity leading to surprising puzzles in two of my favourite games, *The Witness* and *Baba is You*.

Outline

Baba is You

- Sokoban style game (player pushes boxes on a screen).
- Rules are on the screen as sentences, and the player can move along the four cardinal directions to interact with them.



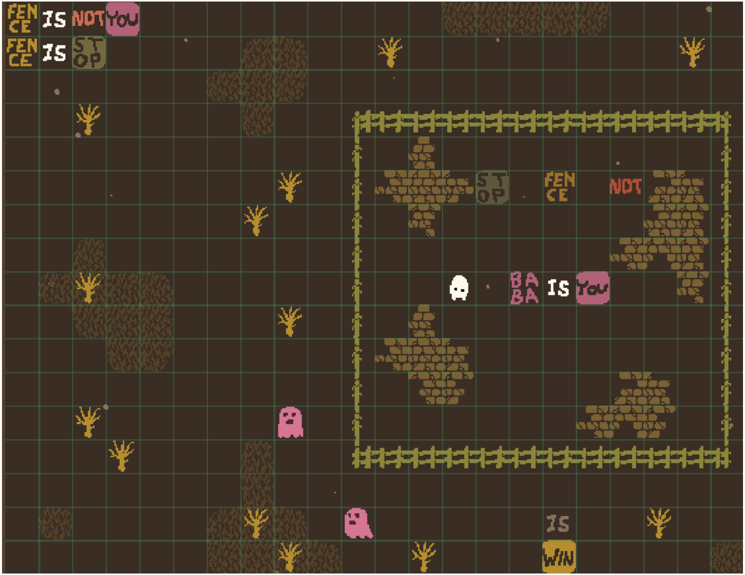
Tutorial Puzzles

- The game never explicitly teaches the players the rules of the game, and instead has tutorial puzzles that introduce each new mechanic.
- "If I exhaustively go through all the meaningful interactions between elements, eventually I get levels where the 'trick' is mostly just the basic functionality of a specific element in itself"- Arvi 'Hempuli' Tekari

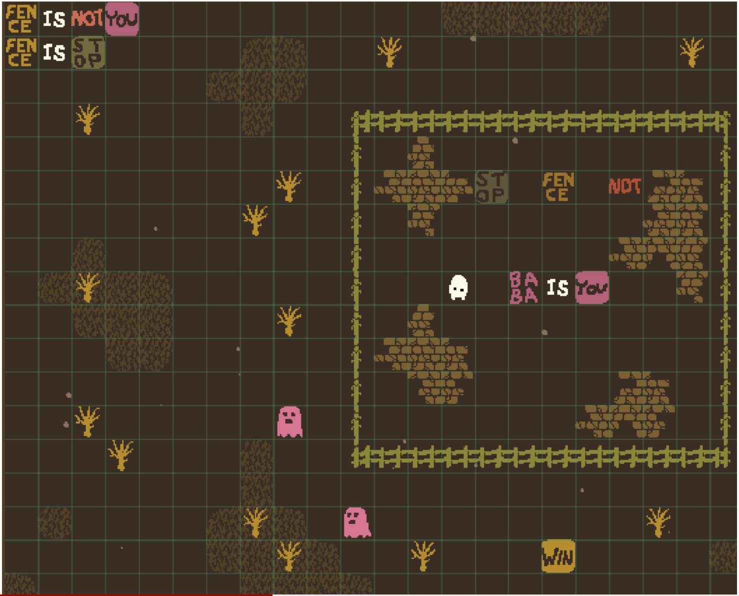
How are puzzles made in this game?

- Game developer Arvi 'Hempuli' Teikari begins by thinking about interesting *interactions* between rules with the huge vocabulary of words with different meanings throughout the game.
- Hempuli then uses *reverse engineering* to come up with a puzzle that forces the user to come up with this clever idea themselves.
- Hempuli does this by *restricting* the player on what they can and can't do in the levels. These restrictions means that the players can only spend so much time exploring ideas that don't work, before they find the idea that does work.

Introduction of Not Rule



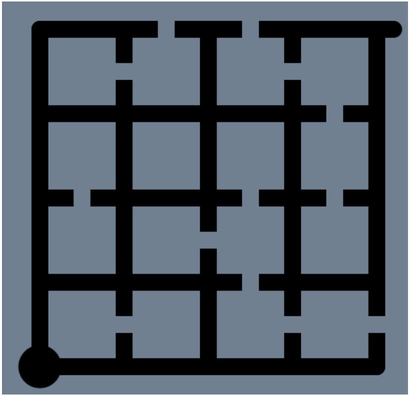
Twist on Not Puzzle



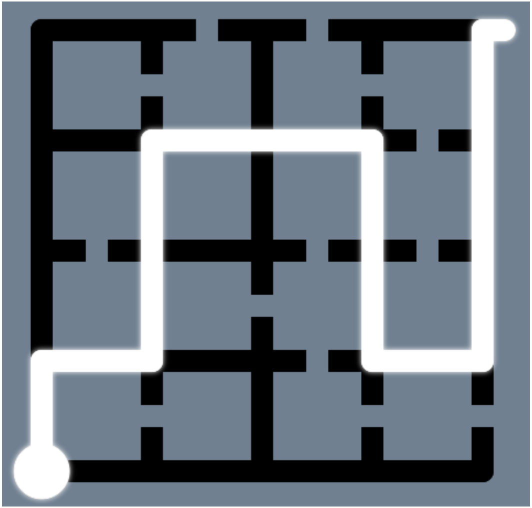
Outline

The Witness

- Beautiful open world puzzle game, where the solution to every puzzle is a path from a start node (the large circle in the bottom left) to the goal node (the half-circle in the top right).
- The grid is 4-connected along the black edges (except for in the cases of broken edges), and the path can never intersect itself.



Previous Puzzle Solution

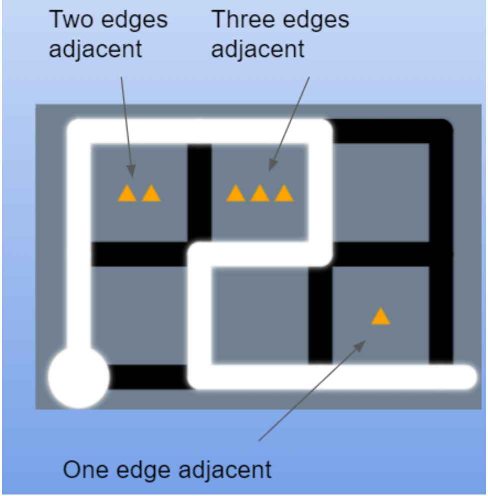


Jonathan Blow Game Design Principles

- Start with an interesting mechanic, in this case, drawing lines from a start node to a goal node.
- Then come up with a rule based on observations from the mechanic in principle. As an example of this, notice that for each gray square in the grid above, a certain number of lines will be adjacent to that square.
- Then think through the consequences of the rule, and what interesting constraints it can create.
- Finally, come up with a puzzle utilizing this rule (possibly) in conjunction with other rules.

Triangle Puzzle Rule

Rule Proposal: The number of triangles in each gray square has to equal the number of edges adjacent to that square.



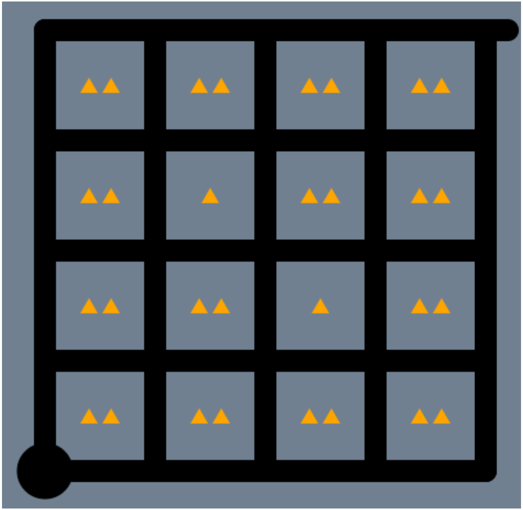
Consequences of Rule

If there's a square with two triangles in a corner, then there's two possible paths through it:

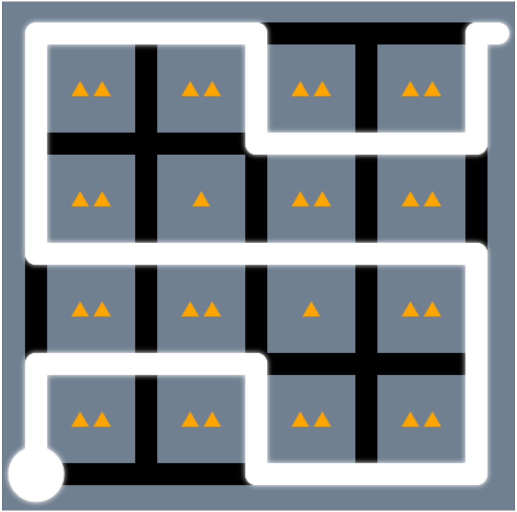
(a) Pattern 1

(b) Pattern 2

Applying this Consequence to a Complicated Puzzle

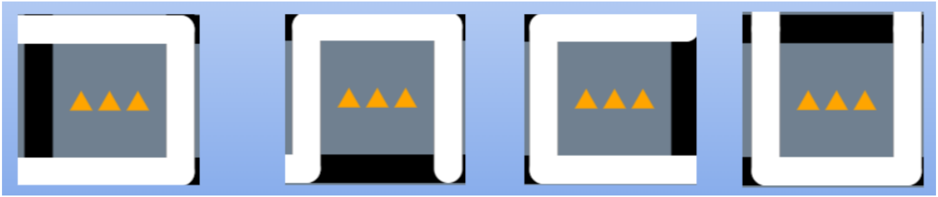


Solution to Complicated Puzzle

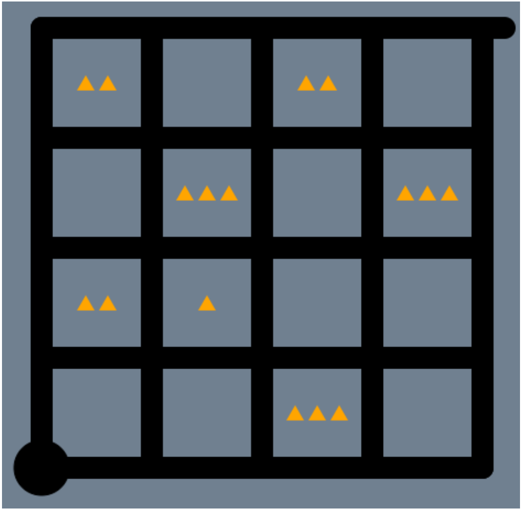


Further Consequences of Triangle Rule

- For any square with 3 triangles in it, there is one of four possible paths.
- These four paths emerge from the no self-intersection rule.



Puzzle Exploring this Consequence



Follow-Up Links

- Stephen Wolfram: A New Kind of Science
- Mandelbrot Set: From Order to Chaos
- Conway's Game of Life Playground
- Game Maker's Toolkit: How Baba Is You Works
- Arvi 'Hempuli' Teikari: Making Baba is You
- Jonathan Blow: Truth in Game Design
- Game Maker's Toolkit: How Jonathan Blow Designs a Puzzle
- The Witness Puzzle Builder